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ROOTS OF UNITY AND THE ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR FORMAL A-MODULES

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ABSTRACT. The cohomology of a Hopf algebroid related to the Adams-Novikov spectral sequence for formal A-modules is studied in the special case in which A is the ring of integers in the field obtained by adjoining pth roots of unity to $\widehat{\mathbb{Q}}_p$, the p-adic numbers. Information about these cohomology groups is used to give new proofs of results about the E_2 term of the Adams spectral sequence based on 2-local complex K-theory, and about the odd primary Kervaire invariant elements in the usual Adams-Novikov spectral sequence.

One of the most powerful tools used in the computation of stable homotopy groups is the Adams-Novikov spectral sequence. The E_2 term of this spectral sequence is a certain Ext group derived from a universal formal group law. In [R3] the corresponding Ext group for a universal formal A-module, for A the ring of algebraic integers in an algebraic number field, K, or its p-adic completion, was introduced and certain conjectures about these groups were formulated. One of these conjectures (concerning the value of $\mathrm{Ext}^{1,*}$) was confirmed in [J] using a Hopf algebroid (i.e., a generalized Hopf algebra in which the left and right units need not agree), E_AT, which generalizes the Hopf algebroid K_*K of stable cooperations for complex K-theory. The present paper is concerned with the cohomology of E_AT in the special case of $A = \widehat{\mathbb{Z}}_p[\zeta]$ where ζ is a pth root of unity and $\widehat{\mathbb{Z}}_p$ denotes the p-adic integers. We will show that in this case E_AT is contained in an extension of Hopf algebroids

$$\widetilde{E_AT} \xrightarrow{j} E_AT \xrightarrow{\rho} \overline{E_AT}$$

and that the cohomology of $\overline{E_AT}$ can be completely described. This provides us with information about the cohomology of E_AT via the Cartan-Eilenberg spectral sequence associated to this extension.

Two applications of this result are presented. In the case $p = 2$, E_AT can be identified with the 2-adic completion of the Hopf algebroid $K_*K_{(2)}$ of stable cooperations for 2-primary complex K-theory. In this case the cohomology

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of $\widetilde{E_A T}$ can also be described, so that we can completely describe the Cartan-Eilenberg spectral sequence (there are no nontrivial differentials for dimensional reasons). We therefore obtain a new proof of the results in [R1, K], computing $H^*(K_*K_{(2)})$.

A second application is the construction of nontrivial elements in the classical Adams-Novikov spectral sequence based on BP, the Brown-Peterson spectrum, which is a summand of MU, the spectrum representing complex cobordism. In [R3] a map of Hopf algebroids

$$\Psi: VT = BP_*BP \to V_A T$$

was described (here $V_A T$ is the Hopf algebroid generalizing VT to the category of formal A-modules, and is constructed using isomorphisms of A-typical formal A-modules). Composing Ψ with the Conner-Floyd map

$$\Phi: V_A T \to E_A T$$

constructed in [J] and the map ρ, we have a map

$$\chi: VT \to \overline{E_A T}$$

from VT to a Hopf algebroid whose cohomology is known. We thus have a tool for identifying nonzero elements of $H^{**}VT$, the E_2 term of the classical Adams-Novikov spectral sequence. We apply this to give a new proof of Theorem 4 of [R1] concerning the odd primary Kervaire invariant elements.

1. An extension containing $E_A T$

In this section, we recall the definition and some of the structure of $E_A T$. We describe the homogeneous components of $E_A T$ and construct two related Hopf algebroids, $\widehat{C}_n$ and $\overline{C}_n$, with which we construct the extension described in the introduction. We conclude by computing the cohomology of $\overline{C}_n$.

The ring $A = \widehat{\mathbb{Z}}_p[\zeta]$ is the ring of integers in the field $K = \widehat{\mathbb{Q}}_p[\zeta]$, which is an extension of $\widehat{\mathbb{Q}}_p$ of degree $p-1$. A has a unique prime ideal (π) whose generator may be taken to be $\pi = \zeta - 1$, and the residue field of A, i.e., $A/(\pi)$, is $\mathbb{Z}/p\mathbb{Z}$. p is totally ramified in A, with $(\pi)^{p-1} = (p)$.

Recall from [J] that the Hopf algebroid $(E_A, E_A T)$ is defined by

$$E_A = A[t, t^{-1}],$$

$$E_A T = \{f \in K[u, u^{-1}, v, v^{-1}] | f(at, bt) \in E_A, \text{ if } a, b \in A, a, b \equiv 1 \ (\pi)\}$$

and that E_A, $E_A T$ are graded with $\deg(t) = \deg(u) = \deg(v) = 2(p-1)$. The structure maps for $(E_A, E_A T)$ are:

$$\begin{aligned} \eta_L(t) &= u, & \eta_R(t) &= v, \\ \psi(u) &= u \otimes 1, & \psi(v) &= 1 \otimes v, \\ c(u) &= v, & c(v) &= u, \\ \varepsilon(u) &= t, & \varepsilon(v) &= t. \end{aligned}$$

If we denote the homogeneous component of $E_A T$ of degree $2 \cdot n \cdot (p-1)$ by $(E_A T)_n$ then we obtain a Hopf algebroid $(A, (E_A T)_n)$. Let us also define

$$C_n = \{f \in K[w, w^{-1}] \| f(a) \in A \text{ if } a \in A, a \equiv 1 \ (\pi)\}.$$

(A, C_n) can be given the structure of a Hopf algebroid via the maps

$$\eta_L(1) = 1, \quad \eta_R(1) = w^n,$$
$$\psi(w) = w \otimes w, \quad c(w) = w^{-1}, \quad \varepsilon(w) = 1.$$

We may define a map $C_n \to (E_A T)_n$ by $f \mapsto u^n \cdot f(v/u)$ and it is straightforward to check that this defines an isomorphism of Hopf algebroids. Thus, in particular we have

$$H^{s, 2n\cdot(p-1)}(E_A T) \simeq H^s(C_n).$$

We will do most of our computations using C_n rather than $E_A T$, and write C in place of C_n if the choice of right unit is not relevant.

Let us also write $B = C \cap K[w]$. We may define a sequence of polynomials in B inductively by

$$q_0 = (w-1)/\pi, \quad q_{i+1} = (q_i^q - q_i)/\pi.$$

Also, let us denote

$$q^I(w) = q_0^{i_0} \cdots q_m^{i_m}$$

if $I = (i_0, \dots, i_m)$ is a multi-index.

Lemma 1. *The polynomials* $\{q^I \mid 0 \le i_j < p, \ m = 0, 1, 2, \dots\}$ *form a basis for* B *as an* A*-module.*

Proof. This is Proposition 7 of [J] (note that these polynomials are denoted there by f_i).

Corollary 2. *The polynomials* $\{q_i | i = 0, 1, \dots\}$ *generate* C *over* $A[w, w^{-1}]$.

It will be useful for us to have a slightly different generating set for C available in addition to this one. Define inductively

$$\tilde{q}_0 = (w-1)/\pi, \quad \tilde{q}_1 = (w^p - 1)/\pi^{p+1}, \quad \tilde{q}_{i+1} = (\tilde{q}_i^p - \tilde{q}_i)/\pi$$

and

$$\tilde{q}^I = \tilde{q}_0^{i_0} \cdots \tilde{q}_m^{i_m} \quad \text{if } I = (i_0, \dots, i_m).$$

Lemma 3. *The polynomials* $\{\tilde{q}^I | 0 \le i_j < p, m = 0, 1, 2, \dots\}$ *form a basis for* B *as an* A*-module.*

Proof. Part of this lemma is, of course, that $\tilde{q}_i(w) \in B$. Since for any $a \in A$, $a^p - a \equiv O(\pi)$, it is sufficient for us to show that $\tilde{q}_1(w) \in B$. This, however, follows from [J, Lemma 17].

To see that this set forms a basis, note that the $(n+1) \times (n+1)$ matrix that expresses the polynomials $\tilde{q}^I$ with $\sum i_j p^j \le n$ as a linear combination of the polynomials q^I is triangular, with diagonal entries equal to 1. Thus, it is invertible over A, and so the polynomials $\tilde{q}^I$ span B. They are clearly linearly independent.

Corollary 4. *The polynomials* $\{\tilde{q}_i | i = 0, 1, \dots\}$ *generate* C *over* $A[w, w^{-1}]$.

Our interest in this second generating set is motivated by the fact, easily proved by induction, that $\tilde{q}_i(w)$ for $i \geq 1$ is a polynomial in w^p. If we denote $\widetilde{C} = C \cap K[w^p, w^{-p}]$ and $\widetilde{B} = \widetilde{C} \cap K[w^p]$, then we have

Corollary 5. *The polynomials* $\{\tilde{q}^I | i_0 = 0, 0 \leq i_j < p, m = 1, 2, \dots\}$ *form a basis for* $\widetilde{B}$ *as an* A*-module.*

Corollary 6. *The polynomials* $\{\tilde{q}_i | i = 1, 2, \dots\}$ *generate* $\widetilde{C}$ *over* $A[w^p, w^{-p}]$.

A third algebra related to C and $\widetilde{C}$ is

$$\widehat{C} = \{f \in K[x, x^{-1}] | f(a) \in A \text{ if } a \equiv 1 \ (\pi^{p+1})\}.$$

We make $(A, \widehat{C}_n)$ into a Hopf algebroid by defining

$$\eta_L(1) = 1, \quad \eta_R(1) = x^n,$$
$$\psi(x) = x \otimes x, \quad c(x) = x^{-1}, \quad \varepsilon(x) = 1.$$

We also define $\widehat{B} = \widehat{C} \cap K[x]$.

The analogs of the polynomials q_i and $\tilde{q}_i$ in this case are the polynomials defined by

$$\hat{q}_1 = (x-1)/\pi^{p+1}, \quad \hat{q}_{i+1} = (\hat{q}_i^p - \hat{q}_i)/\pi.$$

We also use the notation $\hat{q}^I = \hat{q}_1^{i_1} \cdots q_m^{i_m}$ if $I = (i_1, \dots, i_m)$. The analog of Lemmas 1 and 3 is

Lemma 7. *The polynomials* $\{\hat{q}^I | 0 \leq i_j < p, m = 1, 2, \dots\}$ *form a basis for* $\widehat{B}$ *as an* A*-module.*

Proof. The map $K[x] \to K[x]$ defined by $g(x) \mapsto g((x-1)/\pi^{p+1})$ maps the algebra of polynomials with the property that $g(a) \in A$ if $a \in A$ isomorphically to $\widehat{B}$. Since it also maps the basis for this former algebra constructed in [J, Proposition 7], onto the set $\{\hat{q}^I\}$, the latter must be a basis for $\widehat{B}$.

Corollary 8. *The polynomials* $\{\hat{q}_i | i = 1, 2, \dots\}$ *generate* $\widehat{C}$ *over* $A[x, x^{-1}]$.

The connection between $\widehat{C}$ and the previous two Hopf algebras we have considered is given by

Proposition 9. *The map from* $\widehat{C}$ *to* C *that sends* x *to* w^p *is an injection of Hopf algebroids whose image is* $\widetilde{C}$.

Proof. Since this map sends $\hat{q}_i$ to $\tilde{q}_i$, the result is clear.

We next describe the Hopf algebroid $(\overline{E}_A, \overline{E}_A\overline{T})$, or rather we describe its homogeneous, degree $n \cdot 2 \cdot (p-1)$ component, $\overline{C}_n$. Let $\overline{C}_n$ denote the dual of the group algebra of the cyclic group of order p:

$$\overline{C}_n = A[\mathbb{Z}/p\mathbb{Z}]^* = \operatorname{Hom}_A(A[\mathbb{Z}/p\mathbb{Z}], A).$$

The structure maps for $\overline{C}_n$ are, using δ to denote a generator for $\mathbb{Z}/p\mathbb{Z}$,

$$\psi(f)(\delta^i \otimes \delta^j) = f(\delta^{i+j}),$$
$$\eta_L(1)(\delta^i) = 1, \quad \eta_R(1)(\delta^i) = \zeta^{n\cdot i},$$
$$c(f)(\delta^i) = f(\delta^{-i}), \quad \varepsilon(f) = f(1).$$

Let us also define a map of Hopf algebroids $\rho\colon (A, C_n) \to (A, \overline{C}_n)$ by $\rho(f)(\delta^i) = f(\zeta^i)$.

The critical fact about ρ is

Lemma 10. *ρ is a normal map of Hopf algebroids.*

Proof. It is straightforward that ρ preserves the Hopf algebroid structure maps and so defines a map of Hopf algebroids; the question is whether it is normal. Referring to [R4, A1.1.10] we must verify that

$$C_n \,\square_{\overline{C}'_n}\, A = A \,\square_{\overline{C}'_n}\, C_n$$

where $\square$ denotes the cotensor product and, for (A, Γ) a Hopf algebroid, Γ' is the associated Hopf algebra, defined by

$$\Gamma' = \Gamma/(\eta_R(a) - \eta_L(a) | a \in A).$$

In the case $\Gamma = \overline{C}_n$, this becomes

$$\overline{C}'_n = \begin{cases} \overline{C}_n & \text{if } p|n, \\ A & \text{if } (p, n) = 1. \end{cases}$$

To see this, note that if $p|n$, then

$$\begin{aligned}(\eta_R(a) - \eta_L(a))(\delta^i) &= a \cdot (\eta_R(1) - \eta_L(1))(\delta^i) \\ &= a \cdot (\zeta^{n\cdot i} - 1) = a \cdot (1 - 1) = 0\end{aligned}$$

while if $(n, p) = 1$, then the ideal generated by $\eta_R(a) - \eta_L(a)$ is

$$I = \{\phi \in \overline{C}_n | \phi(1) = 0\}.$$

Thus, the map $\overline{C}'_n = \overline{C}_n/I \to A$ that sends ϕ to $\phi(1)$ is an isomorphism.

Since $C_n \square_A A = A \square_A C_n = C_n$, we may assume that $p|n$. The cotensor product $C_n \square_{\overline{C}_n} A$ is defined to be the kernel of the map

$$C_n \simeq C_n \otimes_A A \to C_n \otimes_A \overline{C}_n \otimes_A A$$

which sends f to $(1 \otimes \rho)(\psi f) \otimes 1 - f \otimes \eta_L(1) \otimes 1$. This kernel consists of those elements $f \in \overline{C}_n$ for which

$$(1 \otimes \rho)(f(w \otimes w)) = f(w) \otimes 1$$

in $C_n \otimes_A \overline{C}_n$. These are precisely those elements $f \in \overline{C}_n$ of the form $f(w) = g(w^p)$. Similarly, $A \square_{\overline{C}_n} C_n$ consists of those $f \in C_n$ for which

$$(\rho \otimes 1)(f(w \otimes w)) = \eta_R(1) \otimes f(w)$$

in $\overline{C}_n \otimes C_n$. Since $\eta_R(1) = 1$ in $\overline{C}_n$ when $p|n$, we see that this also consists of those $f \in \overline{C}_n$ of the form $f(w) = g(w^p)$.

If we define the sub-Hopf algebroid $(\tilde{\tilde{A}}, \tilde{\tilde{C}}_n)$ of (A, C_n) by

$$\tilde{\tilde{A}} = A \square_{\overline{C}'_n} A, \quad \tilde{\tilde{C}}_n = A \square_{\overline{C}_n} C_n \square_{\overline{C}_n} A,$$

then, following [R4, A1.1.15], we have

Corollary 11. $(\tilde{\tilde{A}}, \tilde{\tilde{C}}_n) \xrightarrow{i} (A, C_n) \xrightarrow{\rho} (A, \overline{C}_n)$ *is an extension of Hopf algebroids.* (*The fact that* i *is an inclusion is* [R4, A1.1.14].)

For this to be useful we must describe $(\tilde{\tilde{A}}, \tilde{\tilde{C}}_n)$. As noted in the proof of [R4, A1.1.14], we have

$$\tilde{\tilde{A}} = \{a \in A | \eta_L(a) = \eta_R(a) \text{ in } \overline{C}_n\},$$
$$\tilde{\tilde{C}}_n = \{f \in C_n | (\rho \otimes 1 \otimes \rho)\psi^2 f = \eta_L(1) \otimes f \otimes \eta_R(1)\} .$$

and so

$$(\tilde{\tilde{A}}, \tilde{\tilde{C}}_n) = \begin{cases} 0 & \text{if } (n, p) = 1, \\ (A, \widetilde{C}_n) \simeq (A, \widehat{C}_{n/p}) & \text{if } n|p. \end{cases}$$

The applications we have in mind for this extension involve the cohomology of C_n, which we approach via that of $\widetilde{C}_n$ and $\overline{C}_n$. We conclude this section, therefore, by recalling the cohomology of $\overline{C}_n$. Let us define two homomorphisms $S, T: \overline{C}_n \to \overline{C}_n$ by

$$S(f)(x) = f(\delta x) - f(x) \quad \text{and} \quad T(f)(x) = \sum_{i=0}^{p-1} f(\delta^i x) .$$

A straightforward computation yields

Lemma 12. $0 \to A \xrightarrow{\eta_L} \overline{C}_n \xrightarrow{S} \overline{C}_n \xrightarrow{T} \overline{C}_n \xrightarrow{S} \cdots$ *is an injective resolution of* A *considered as a left* C_n *comodule.*

Corollary 13. *The cohomology of* $\overline{C}_n$ *is given by*

$$H^s(\overline{C}_n) = \begin{cases} A/\pi A, & s \text{ odd}, \\ 0, & s \text{ even}, \end{cases}$$

if $(n, p) = 1$, *and by*

$$H^s(\overline{C}_n) = \begin{cases} A, & s = 0, \\ A/pA, & s > 0,\ s \text{ even}, \\ 0, & s \text{ odd}, \end{cases}$$

if $p|n$.

Proof. Applying the functor $A \square_{C_n} (\ )$ to the resolution of A and using the identification $A \square_{C_n} C_n = A$ gives the complexes

$$A \xrightarrow{\zeta^n - 1} A \xrightarrow{0} A \xrightarrow{\zeta^n - 1} A \xrightarrow{0} \cdots$$

if $(n, p) = 1$, and

$$A \xrightarrow{0} A \xrightarrow{p} A \xrightarrow{0} A \xrightarrow{p} A \longrightarrow \cdots$$

if $p|n$.

2. Applications

2.1. The cohomology of $K_*K_{(2)}$. If the prime p is chosen to be 2, then $A = \widehat{\mathbb{Z}}_2$ and C_n can be described as

$$C_n = \{f \in \widehat{\mathbb{Q}}_2[w, w^{-1}] | f(a) \in \widehat{\mathbb{Z}}_2 \text{ if } a \equiv 1\ (2)\}.$$

The description of K_*K given in [AHS]

$$K_*K = \{f \in \mathbb{Q}[u, u^{-1}, v, v^{-1}] | f(at, bt) \in \mathbb{Z}[t, t^{-1}, 1/a, 1/b] \\ \text{if } a, b \in \mathbb{Z}, a, b \neq 0\}$$

shows that C_n can be identified with $(K_*K_{(2)})_n$ so that the E_2 term of the Adams spectral sequence based on 2-local complex K-theory has as its completion

$$E_2^{*,n} = H^*(C_n) = \mathrm{Ext}^*_{C_n}(\widehat{\mathbb{Z}}_2, \widehat{\mathbb{Z}}_2).$$

The Cartan-Eilenberg spectral sequence, [R4, A1.3.14], allows us to describe these groups in terms of the cohomology of $\widetilde{C}_n$ and $\overline{C}_n$:

Proposition 14. *There is a spectral sequence converging to* $\mathrm{Ext}^*_{C_n}(\widehat{\mathbb{Z}}_2, \widehat{\mathbb{Z}}_2)$ *whose* E_2 *term is* $E_2^{s,t} = \mathrm{Ext}^s_{\widetilde{C}_n}(\widehat{\mathbb{Z}}_2, \mathrm{Ext}^t_{\overline{C}_n}(\widehat{\mathbb{Z}}_2, \widehat{\mathbb{Z}}_2))$.

Since $\mathrm{Ext}^*_{\overline{C}_n}(\widehat{\mathbb{Z}}_2, \widehat{\mathbb{Z}}_2)$ is described at the end of §1, we turn to describing $\mathrm{Ext}^*_{\widetilde{C}_n}(\widehat{\mathbb{Z}}_2, \ )$. The key to this description is the following injective resolution, which is the analog at the prime 2 of a resolution constructed for odd primes in [B, §7].

Lemma 15. *The sequence*

$$0 \longrightarrow \widehat{\mathbb{Z}}_2 \xrightarrow{p_0} \widehat{C}_n \xrightarrow{p_1} \widehat{C}_n \xrightarrow{p_2} \widehat{\mathbb{Q}}_2 \longrightarrow 0$$

defined by $p_1(f) = f(9w) - f(w)$ *and* $p_2(\sum a_i w^i) = a_0$ *is an injective resolution of* $\widehat{\mathbb{Z}}_2$.

(*The left* $\widehat{C}_n$ *comodule structure of* $\widehat{\mathbb{Z}}_2$ *and* $\widehat{\mathbb{Q}}_2$ *is that defined by* η_L.)

(*The factor* $9 = 2^3 + 1$ *occurs here because it is a generator of*

$$(1 + 2^3\widehat{\mathbb{Z}}_2)/(1 + 2^n\widehat{\mathbb{Z}}_2)$$

for $n \geq 4$.)

Proof. If $p_1(\sum a_i w^i) = \sum_i a_i(9^i - 1) \cdot w^i = 0$ then $a_i = 0$ for $i \neq 0$ and the integrality condition for $\widehat{C}_n$ shows that $a_0 \in \widehat{\mathbb{Z}}_2$. Thus, $\ker(p_1) = \mathrm{Im}(p_0)$.

The fact that the polynomials $(w^{2^k}-1)/2^{k+3}$ are in $\widehat{C}_n$ shows that p_2 is surjective.

It remains to verify that $\ker(p_2)=\operatorname{Im}(p_1)$. Suppose that

$$f=\sum_{i\neq 0} a_i w^i \in \ker(p_2).$$

For any $a\in\widehat{\mathbb{Q}}_2$, the polynomial

$$g(w)=a+\sum_{i\neq 0}\frac{a_i w^i}{9^i-1}$$

is mapped to f by p_1. The question is whether a can be chosen so that $g\in\widehat{C}_n$. Choose a so that $g(1)=0$. Since $p_1(g)=g(9w)-g(w)\in\widehat{C}_n$ it follows by induction on k that $g(9^k)\in A$ for any k, and this is enough to imply $g\in\widehat{C}_n$. To see this, first note that there exists m such that $2^m\cdot g\in\widehat{\mathbb{Z}}_2[w,w^{-1}]$, and such that if $a, b\in\widehat{\mathbb{Z}}_2$, then $g(b)\in\widehat{\mathbb{Z}}_2$. However, $(1+2^3\widehat{\mathbb{Z}}_2)/(1+2^m\widehat{\mathbb{Z}}_2)$ is cyclic, generated by 9. Thus, if $a\in 1+2^3\widehat{\mathbb{Z}}_2$, then $a\equiv 9^k \bmod 2^m$ for some k and so $g(a)\in\widehat{\mathbb{Z}}_2$.

Corollary 16.

(a)

$$\operatorname{Ext}^s_{\widehat{C}_n}(\widehat{\mathbb{Z}}_2,\widehat{\mathbb{Z}}_2)=\begin{cases}\widehat{\mathbb{Z}}_2, & s=0,\\ \widehat{\mathbb{Q}}_2/\widehat{\mathbb{Z}}_2, & s=2,\\ 0, & \textit{otherwise};\end{cases}$$

(b) *for* $n\neq 0$

$$\operatorname{Ext}^s_{\widehat{C}_n}(\widehat{\mathbb{Z}}_2,\widehat{\mathbb{Z}}_2)=\begin{cases}\mathbb{Z}/2^{d(n)}\mathbb{Z}, & s=1,\\ 0, & \textit{otherwise};\end{cases}$$

(c)

$$\operatorname{Ext}^s_{\widehat{C}_n}(\widehat{\mathbb{Z}}_2,\mathbb{Z}/2\mathbb{Z})=\begin{cases}\mathbb{Z}/2\mathbb{Z}, & s=0,1,\\ 0, & \textit{otherwise.}\end{cases}$$

Here $d(n)$ is the largest integer such that $2^{d(n)}$ divides $2^3\cdot n$.

Proof. $\operatorname{Ext}_{\widehat{C}_n}(\widehat{\mathbb{Z}}_2,\widehat{\mathbb{Z}}_2)$ is the cohomology of the complex

$$\widehat{\mathbb{Z}}_2\square_{\widehat{C}_n}\widehat{C}_n\to\widehat{\mathbb{Z}}_2\square_{\widehat{C}_n}\widehat{C}_n\to\widehat{\mathbb{Z}}_2\square_{\widehat{C}_n}\widehat{\mathbb{Q}}_2\to 0$$

If $n=0$ this complex is

$$\widehat{\mathbb{Z}}_2\xrightarrow{0}\widehat{\mathbb{Z}}_2\xrightarrow{1}\widehat{\mathbb{Q}}_2\longrightarrow 0$$

and, if $n\neq 0$

$$\widehat{\mathbb{Z}}_2\xrightarrow{9^n-1}\widehat{\mathbb{Z}}_2\longrightarrow 0\longrightarrow 0.$$

These account for (a) and (b), since the highest power of 2 dividing $9^n - 1$ is $2^{d(n)}$. For (c), we are interested in the cohomology of

$$\widehat{\mathbb{Z}}_2 \square_{\widehat{C}_n} \widehat{C}_n \otimes_A \mathbb{Z}/2\mathbb{Z} \to \widehat{\mathbb{Z}}_2 \square_{\widehat{C}_n} \widehat{C}_n \otimes_A \widehat{\mathbb{Z}}/2\mathbb{Z} \to \widehat{\mathbb{Z}}_2 \square_{\widehat{C}_n} \otimes_A \mathbb{Z}/2\mathbb{Z} \to 0.$$

This complex is, for any n,

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Combining these results with Proposition 14 and Corollary 13, we obtain

Corollary 17. *The E_2 term of the spectral sequence of Proposition* 14 *is*

$$E_2^{s,t} = \begin{cases} \widehat{\mathbb{Z}}_2, & \textit{if } (s,t) = (0,0),\ n = 0, \\ \widehat{\mathbb{Q}}_2/\widehat{\mathbb{Z}}_2, & \textit{if } (s,t) = (2,0),\ n = 0, \\ \mathbb{Z}/2^{d(m)}\mathbb{Z}, & \textit{if } (s,t) = (1,0),\ n = 2m = 0, \\ \mathbb{Z}/2\mathbb{Z}, & \textit{if } (s,t) = (0,2t'),\ n = 2m, \textit{ or } (1,2t'), \\ 0, & \textit{otherwise}. \end{cases}$$

Corollary 18.

$$\operatorname{Ext}^{s,t}_{K_*K_{(2)}}(\pi_*K, \pi_*K) = \begin{cases} \mathbb{Z}_{(2)}, & \textit{if } (s,t) = (0,0), \\ \mathbb{Z}/2^{\infty}\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \textit{if } (s,t) = (2,0), \\ \mathbb{Z}/2^{d(m)}\mathbb{Z}, & \textit{if } (s,t) = (1,2m) \neq (1,0), \\ \mathbb{Z}/2\mathbb{Z}, & \textit{if } (s,t) = (s,2t') \neq (2,0),\ s \geq 2, \\ 0, & \textit{otherwise}. \end{cases}$$

2.2. The odd primary Kervaire invariant elements. The Hopf algebroid (V_A, V_AT) is constructed using isomorphisms of A-typical formal A-modules. If $A = \mathbb{Z}_{(p)}$, then one obtains $(V, VT) = (BP_*, BP_*BP)$, the Hopf algebroid of Brown-Peterson homology. If A is a $\mathbb{Z}_{(p)}$ algebra as in the case $A = \widehat{\mathbb{Z}}_p[\zeta]$ with which we are concerned, then a formal A-module is also a formal $\mathbb{Z}_{(p)}$ module. Thus, we obtain, as in [R3, 3.11], as map of Hopf algebroids

$$\Psi\colon (V, VT) \to (V_A, V_AT).$$

Composing this with the generalized Conner-Floyd map

$$\Phi\colon (V_A, V_AT) \to (E_A, E_AT)$$

of [J] and with $\rho\colon (E_A, E_AT) \to (E_A, \overline{E_AT})$ we obtain a map

$$\chi\colon (V, VT) \to (E_A, \overline{E_AT})$$

and so a map in cohomology

$$\chi^*\colon H^*VT \to H^*(E_AT).$$

We will show that a family of interesting elements in H^*VT, the odd primary Kervaire invariant elements, have nonzero image under this map.

Recall that (V, VT) has the description

$$V = \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \qquad VT = V[t_1, t_2, \ldots],$$

and that V, VT are graded with $\deg(v_i) = \deg(t_i) = 2(p^i - 1)$. The elements $h_0, b_i \in H^{1, 2(p-1)}(VT), H^{2, 2(p-1)p^{i+1}}(VT)$, respectively, are represented in the cobar complex of VT by $h_0 = [t_1]$ and

$$b_i = \frac{1}{p} \sum_{j=1}^{p^{i+1}-1} (p_j^{i+1})[t_1^j \otimes t_1^{p^{i+1}-j}].$$

Our result is

Proposition 19. *All monomials in* h_0, b_i, $i = 0, 1, 2, \ldots$, *have nonzero image in* $H^*(E_A T)$ *under* χ^*.

Proof. It is straightforward to describe the map of cobar complexes induced by χ. We also need, however, a method of identifying cohomologically nontrivial elements in the cobar complex of $E_A T$ or $\overline{C}_n$. For this we define a chain map from the cobar complex of $\overline{C}_n$ to the complex described in §1, Lemma 12.

Recall from [R4, A1.2.11] that the cobar resolution of A as a $\overline{C}_n$ comodule has as its sth term $\overline{C}_n \otimes (\ker(\varepsilon))^{\otimes s}$ and that the differential is given by

$$d(\gamma_0 \otimes \cdots \otimes \gamma_s) = \sum_{i=0}^{s} (-1)^s \gamma_0 \otimes \cdots \otimes \psi(\gamma_i) \otimes \cdots \otimes \gamma_s + (-1)^{s+1} \gamma_0 \otimes \cdots \otimes \gamma_s.$$

If we identify elements of

$$\overline{C}_n \otimes (\ker(\varepsilon))^{\otimes s} \subseteq \overline{C}_n^{\otimes s+1} = \mathrm{Hom}_A(A[\mathbb{Z}/p\mathbb{Z}], A)^{\otimes s+1}$$

with multilinear maps from $A[\mathbb{Z}/p\mathbb{Z}]^{s+1}$ to A, then the differential becomes

$$\begin{aligned} df(w_0, \ldots, w_{s+1}) = \sum_{i=0}^{s} (-1)^i f(w_0, \ldots, w_i \cdot w_{i+1}, \ldots, w_{s+1}) \\ + (-1)^{s+1} f(w_0, \ldots, w_s). \end{aligned}$$

Using this identification we define a chain map, R, from the cobar resolution of A over $\overline{C}_n$ to the resolution described in Lemma 12.

$$R(f)(w) = \begin{cases} f(w, \zeta) & \text{if } s = 1, \\ \sum_{i_1, \ldots, i_{s-1/2}=1}^{p-1} f(w, \zeta^{i_1}, \zeta, \zeta^{i_2}, \ldots, \zeta), & s \text{ odd}, \\ \sum_{i_1, \ldots, i_{s/2}=1}^{p-1} f(w, \zeta, \zeta^{i_1}, \zeta, \zeta^{i_2}, \ldots, \zeta^{i_{s/2}}), & s \text{ even}. \end{cases}$$

Applying $A\square_{C_n}(\ )$ we obtain a map from the cobar complex of $\overline{C}_n$ to the complex of Corollary 13. We denote this map by R as well. It is given by

$$R(f) = \begin{cases} f(\zeta), & s = 1, \\ \sum f(\zeta^{i_1}, \zeta, \zeta^{i_2}, \ldots, \zeta), & s \text{ odd}, \\ \sum f(\zeta^{i_1}, \zeta, \zeta^{i_2}, \ldots, \zeta^{i_{s/2}}), & s \text{ even}. \end{cases}$$

Under the composition $\Phi \circ \Psi$, the elements h_0 and b_i are mapped to $(w-1)/\pi$ and

$$\frac{1}{p}\sum_{j=1}^{p^{i+1}-1}\binom{p^{i+1}}{j}\left(\frac{w-1}{\pi}\right)^j \otimes \left(\frac{w-1}{p}\right)^{p^{i+1}-j}$$

in the cobar complex of C_n. Under the composition $R \circ \rho$, these are mapped to 1 and

$$\frac{1}{p}\sum_{j=1}^{p-1}\left(\left(\frac{\zeta^j-1}{\pi}+1\right)^{p^{i+1}} - \left(\frac{\zeta^j-1}{\pi}\right)^{p^{i+1}} - 1\right),$$

respectively. We denote the latter element of A by k_i. This series of maps will send the monomial $h_0^{\varepsilon} b_1^{i_1} \cdots b_m^{i_m}$ to $k_1^{i_1} \cdots k_m^{i_m}$. Showing that $k_i \not\equiv 0 \bmod \pi$ will, therefore, complete the proof of Proposition 19.

$$\begin{aligned}
&\frac{1}{p}\sum_{j=1}^{p-1}\left(\left(\frac{\zeta^i-1}{\pi}+1\right)^{p^{i+1}} - \left(\frac{\zeta^i-1}{\pi}\right)^{p^{i+1}} - 1\right)\\
&\quad = \frac{1}{p}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1^{i+1}}\binom{p^{i+1}}{k}\left(\frac{\zeta^i-1}{\pi}\right)^k\\
&\quad \equiv \frac{1}{p}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}\binom{p^{m+1}}{k\cdot p^m}\left(\frac{\gamma^{j-1}}{\pi}\right)^{k\cdot p^m} \qquad \bmod \pi\\
&\quad \equiv \frac{1}{p}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}\binom{p}{k}\left(\frac{\gamma^{j-1}}{\pi}\right)^k \qquad \bmod \pi\\
&\quad \equiv \frac{1}{p}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}\binom{p}{k} j^k \qquad \bmod \pi\\
&\quad = \frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k}\sum_{j=1}^{p-1} j^k \equiv \frac{-1}{p}\binom{p}{p-1} \qquad \bmod \pi\\
&\quad \equiv -1 \quad \bmod \pi.
\end{aligned}$$

In these congruences, we have used the fact that $\binom{p^{i+1}}{k}$ is divisible by p^2 unless k is divisible by p^i, that $\binom{p^{i+1}}{kp^i} \equiv \binom{p}{k} \bmod p$, that since $\zeta = \pi + 1$, $((\zeta^i - 1)/\pi) \equiv i \bmod \pi$, and, finally, that $\sum_{j=1}^{p-1} j^k \equiv 0 \bmod p$ if $k < p-1$ and that $\sum_{j=1}^{p-1} j^{p-1} \equiv -1 \bmod p$.

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